Time-Varying Fields and Maxwell's Equations

المجالات المتغيرة مع الزمن و معادلات ماكسويل

Two new concepts will be introduced: the electric field produced by a changing magnetic field and the magnetic field produced by a changing electric field. The first of these concepts resulted from experimental research by Michael Faraday and the second from the theoretical efforts of James Clerk Maxwell.

9.1 FARADAY'S LAW

After Oersted demonstrated in 1820 that an electric current affected a compass needle, Faraday professed his belief that if a current could produce a magnetic field, then a magnetic field should be able to produce a current.

In terms of fields, we now say that a **time-varying magnetic field produces an *electromotive force*** (emf) that may establish a current in a suitable closed circuit. **An electromotive force is merely a voltage that arises from conductors moving in a magnetic field or from changing magnetic fields,** and we shall define it in this section. Faraday's law is customarily stated as

$$emf = -\frac{d\Phi}{dt} \qquad V$$

A nonzero value of $\frac{d\Phi}{dt}$ may result from any of the following situations:

1. A time-changing flux linking a stationary closed path

2. Relative motion between a steady flux and a closed path

3. A combination of the two

If the closed path is that taken by an N-turn filamentary conductor, it is often sufficiently accurate to consider the turns as coincident and let

$$emf = -N\frac{d\Phi}{dt} \qquad V$$

The emf is obviously a scalar, and (perhaps not so obviously) a dimensional check shows that it is measured in volts. We define the emf as

$$emf = \oint E.dL$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

This is one of Maxwell's four equations as written in differential, or point, form, the form in which they are most generally used

9.2 Conductors in Motion Through Time-Independent Fields

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The force F on a charge Q in a magnetic field B, where the charge is moving with velocity U is

$$F = Q(v \times B)$$

Motional electric field intensity, E_m, can be defined as the force per unit charge:

$$E_m = \frac{F}{Q} = (v \times B)$$

When a conductor with a great number of free charges moves through a field B, the impressed E_m creates a voltage difference between the two ends of the conductor, the magnitude of which depends on how E_m is oriented with respect to the conductor. With conductor ends a and b, the voltage of **a** with respect to **b** is

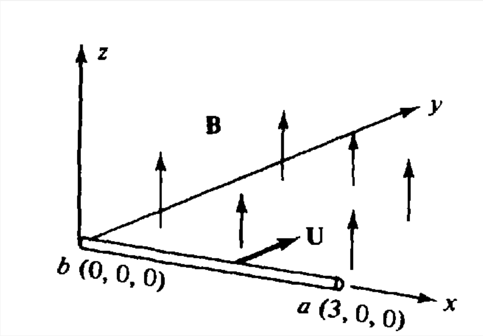
$$V_{ab} = \int_a^b E_m.dL = \int_a^b (v \times B).dL$$

If the velocity U and the field B are at right angles, and the conductor is normal to both, then a conductor of length *l* will have a voltage

$$V = Blv$$

Example: In Figure below, two conducting bars move outward with velocities $v_1 = -12.5\mathbf{a}_y$ m/s and $v_2 = 8\mathbf{a}_y$ m/s in the field **B** = **0.35a$_z$** T. Find the voltage V_{ab} *and* V_{cd}?

Solution:



$$E_{m1} = v_1 \times B = -12.5\mathrm{a}_y \times 0.35\mathrm{a}_z = -4.375\mathrm{a}_x$$

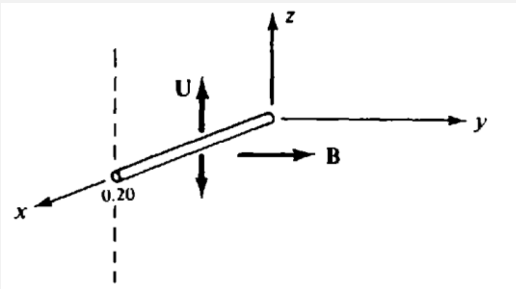
$$E_{m1} = v_2 \times B = 8\mathbf{a}_y \times 0.35\mathrm{a}_z = 2.8\ \mathrm{a}_x$$

$$V_{ab} = \int_a^b E_{m1} \,.\, dL_1 = \int_0^{0.5} -4.375\mathrm{a}_x \,.\, dx\mathrm{a}_x = -2.19\ V$$

$$V_{cd} = \int_c^d E_{m2} \,.\, dL_2 = \int_0^{0.5} 2.8\ \mathrm{a}_x \,.\, dx\mathrm{a}_x = 1.4\ V$$

Example: Find the induced voltage in the conductor of Figure below where **B** = **0.04a$_y$** T and $\boldsymbol{v = 2.5 \sin 10^3\, t}$ m/s?

Solution:



$$E = v \times B$$

$$= 2.5 \sin 10^3\, t \times 0.04\mathrm{a}_y$$

$$= -0.1 \sin 10^3\, t\ \mathrm{a}_x$$

$$V = \int_a^b E \,.\, dL = \int_0^{0.5} -0.1 \sin 10^3\, t\ \mathrm{a}_x \,.\, dx\mathrm{a}_x$$

$$= -0.02 \sin 10^3\, t \quad V$$

9.3 Conductors in Motion Through Time-Dependent Fields

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When a closed conducting loop is ***in motion*** and also the field **B** is a ***function of time***, then the total induced voltage is made up of a contribution from each of the two sources of flux change. Faraday's law becomes

$$V = -\int_S \frac{\partial B}{\partial t} . dS + \oint (v \times B) . dL$$

The first term on the right is the voltage due to the change in B, with the loop held fixed; the second term is the voltage arising from the motion of the loop, with B held fixed.

Example: An area of 0.65 m^2 in the z = 0 plane is enclosed by a filamentary conductor. Find the induced voltage, if $B = 0.05 \cos 10^3\, t \ \left(\frac{a_y + a_z}{\sqrt{2}}\right)$?

Solution:

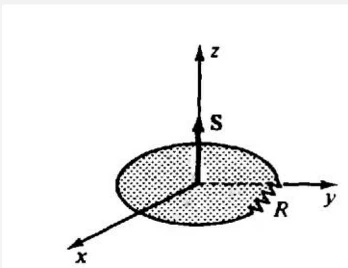
$$V = -\int_S \frac{\partial B}{\partial t} . dS = -\int_S \frac{\partial}{\partial t}\left(0.05 \cos 10^3\, t \ \left(\frac{a_y + a_z}{\sqrt{2}}\right)\right) . dS a_z$$

$$V = \int_S 50 \sin 10^3 t \ * \frac{1}{\sqrt{2}} * 0.65$$

$$V = 23 \sin 10^3 t$$

Example: The circular loop conductor lies in the z = 0 plane, has a radius of 0.10 m and a resistance of 5Ω. Given $B = 0.2 \sin 10^3\, t\, a_z$ determine the current?

Solution:



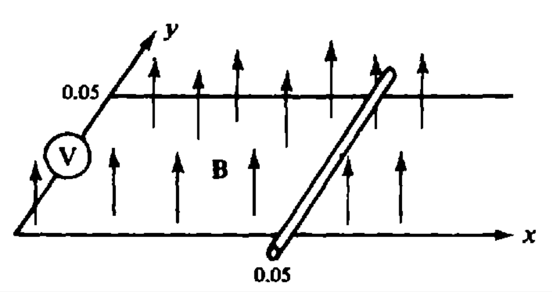
$$V = -\int_S \frac{\partial B}{\partial t} . dS = -\int_0^{2\pi}\int_0^{0.1} \frac{\partial}{\partial t}(0.2 \sin 10^3\, t\, a_z) . \rho d\rho d\emptyset a_z$$

$$V = -200 \cos 10^3 t \ * 2\pi * \left(\frac{0.1^2}{2}\right)^2 = -2\pi \cos 10^3 t$$

$$i = \frac{V}{R} = \frac{-2\pi \cos 10^3 t}{5} = -0.4\pi \cos 10^3 t$$

Example: The bar conductor parallel to the y axis shown in Fig. below completes a loop by sliding contact with the conductors at y = 0 and y = 0.05 m. (a) Find the induced voltage when the bar is stationary at x = 0.05 m and $B = 0.3 \sin 10^4\, t\, \mathrm{a}_z$ T. (b) Repeat for a velocity of the bar $v = 150\mathbf{a}_x$ m/s?

Solution:



(a)

$$V = -\int_S \frac{\partial B}{\partial t} \cdot dS$$

$$= -\int_0^{0.05}\int_0^{0.05} \frac{\partial}{\partial t}(0.3 \sin 10^4\, t\, \mathrm{a}_z).\, dxdy\mathrm{a}_z$$

$$= -\int_0^{0.05}\int_0^{0.05} 3000 \cos 10^4 t\ dxdy = -7.5 \cos 10^4 t \quad V$$

(b)

$$V = -\int_S \frac{\partial B}{\partial t} \cdot dS + \oint (v \times B).\, dL$$

$$(v \times B) = 150\mathbf{a}_x \times 0.3 \sin 10^4\, t\, \mathrm{a}_z = -45 \sin 10^4\, \mathrm{a}_y$$

$$V = -\int_S \frac{\partial B}{\partial t} \cdot dS + \oint (v \times B).\, dL = -7.5 \cos 10^4 t + \int -45 \sin 10^4\, \mathrm{a}_y\ .\, dy\mathrm{a}_y$$

$$V = 7.5 \cos 10^4 t - 2.25 \sin 10^4$$

9.4 Displacement Current تيار الازاحـــــة

In static fields the curl of H was found to be point wise equal to the current density J_c. This is ***conduction current*** density; the subscript c has been added to emphasize that moving charges-electron, photons, or ions—compose the current.

$$\nabla \times H = J_c$$

$$\nabla.(\nabla \times H) = \nabla. J_c$$

$$0 = \nabla. J_c$$

But the continuity equation would be

$$\nabla. J_c = -\frac{\partial \rho}{\partial t}$$

Hence, James Clerk Maxwell postulated that

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \mathbf{J_c} + \mathbf{J_D} = \mathbf{J_c} + \frac{\boldsymbol{\partial D}}{\boldsymbol{\partial t}}$$

Where

$$\mathrm{J_D} = \frac{\partial D}{\partial t} \qquad , \qquad \mathrm{J_c} = \sigma E$$

The displacement current i_D through a specified surface is obtained by integration of the normal component of J_D over the surface

$$i_D = \int_S \mathrm{J_D} \,.\, dS = \int_S \frac{\partial D}{\partial t} . dS$$

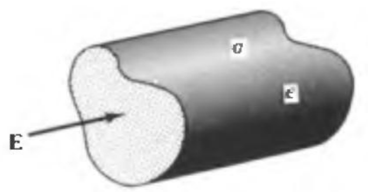
Conduction current density, $\mathrm{J_c} = \sigma E$ is the motion of charge (usually electrons) in a region of zero net charge density, and convection current density,

Ratio of Jc to J_D

Some materials are neither good conductors nor perfect dielectrics, so that both conduction current and displacement current exist. A model for the poor conductor or lossy dielectric is shown in Figure below. Assuming the time dependence $e^{j\omega t}$for E. the total current density is

$$\mathrm{J}_T = \mathrm{J}_c + \mathrm{J}_D = \sigma E + \frac{\partial}{\partial t}\sigma E = \sigma E + j\omega\varepsilon E$$

$$\frac{\mathrm{J}_c}{\mathrm{J}_D} = \frac{\sigma}{\omega\varepsilon}$$



As expected, the displacement current becomes increasingly important as the frequency increases.

Example: A circular-cross-section conductor of radius 1.5 mm carries a current if $i_c = 5.5 \sin 4 \times 10^{10}\ t\ \ \mu A$. What is the amplitude of the displacement current density, if $\sigma = 35\ MS/m$ and $\varepsilon_r = 1$?

Solution:

$$\frac{i_c}{i_D} = \frac{\sigma}{\omega\varepsilon} \quad \Rightarrow\ i_D = \frac{\omega\varepsilon}{\sigma} i_c = \frac{4 \times 10^{10} \times 8.85 \times 10^{10}}{35 \times 10^{10}} \times 5.5 \times 10^{-6} = 55.62 \times 10^{-6}$$

$$J_D = \frac{i_D}{S} = \frac{55.62 \times 10^{-6}}{\pi(1.5 \times 10^{-3})^2} = 7.87 \times 10^{-3}\ \mu A/m^2$$

Example: In a material for which $\sigma = 5$ S/m and $\varepsilon_r = 1$ the electric field intensity is $E = 250 \sin 10^{10}\, t \;\; V/m$. Find (a) the conduction and displacement current densities, and (b) the frequency at which they have equal magnitudes?

Solution:

(a)

$$J_c = \sigma E = 5 * 250 \sin 10^{10}\, t \;\; = 1250 \sin 10^{10}\, t$$

$$J_D = \frac{\partial D}{\partial t} = \frac{\partial}{\partial t}(\varepsilon_r \varepsilon_0 250 \sin 10^{10}\, t\,) = 22.1 \cos 10^{10} t$$

(b)

$$J_c = J_D$$

$$1 = \frac{\sigma}{\omega\varepsilon} \quad \Rightarrow \omega = \frac{\sigma}{\varepsilon} = \frac{5}{8.85 \times 10^{-12}} = 5.65 \times 10^{11}$$

$$f = \frac{\omega}{2\pi} = \frac{5.65 \times 10^{11}}{2\pi} = 89.9\, GHz$$

Example: Find the displacement current density associated with the magnetic field (assume zero conduction current) $H = A1 \sin(4x)\cos(wt - \beta z)\, \mathrm{a}_x + A2\cos(4x)\sin(wt - \beta z)\, \mathrm{a}_z$?

Solution:

$$\nabla \times H = J_c + J_D \qquad , \quad J_c = 0$$

$$J_D = \nabla \times H = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & 0 & H_z \end{vmatrix}$$

$$J_D = \frac{\partial H_z}{\partial y}\mathbf{a}_x + \left[\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right]\mathbf{a}_y - \frac{\partial H_x}{\partial y}\mathbf{a}_z$$

$$J_D = \frac{\partial}{\partial y}(A2\cos(4x)\sin(wt - \beta z))\mathbf{a}_x$$

$$+ \left[\frac{\partial}{\partial z}(A1\sin(4x)\cos(wt - \beta z)) - \frac{\partial}{\partial x}(A2\cos(4x)\sin(wt - \beta z))\right]\mathbf{a}_y$$

$$- \frac{\partial}{\partial y}(A1\sin(4x)\cos(wt - \beta z))\mathbf{a}_z$$

$$J_D = [\beta A1\sin(4x)\sin(wt - \beta z) + 4A2\sin(4x)\sin(wt - \beta z)]\ \mathbf{a}_y$$

$$J_D = (4A2 + \beta A1)\sin(4x)\sin(wt - \beta z)\ \mathbf{a}_y$$

9.5 Maxwell's Equations in Point Form معادلات ماكسويل للمجالات المتغيرة مع الزمن

A static E field can exist in the absence of a magnetic field H; a capacitor with a static charge Q furnishes an example. Likewise, a conductor with a constant current I has a magnetic field H without an E field. When fields are time-variable, however, H ***cannot exist*** without an E field nor can E exist without a corresponding H field.

The equations grouped below, called Maxwell's equations in time-varying fields

$$\nabla \times E = -\frac{\partial B}{\partial t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\nabla \times H = \mathrm{J} + \frac{\partial D}{\partial t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\nabla . \mathrm{D} = \rho_{\mathrm{v}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$\nabla . \mathrm{B} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Equation (3) essentially states that charge density is a source (or sink) of electric flux lines. Equation (4) again acknowledges the fact that "magnetic charges," or poles, are not known to exist. Magnetic flux is always found in closed loops and never diverges from a point source.

These four equations form the basis of all electromagnetic theory. They are partial differential equations and relate the electric and magnetic fields to each other and to their sources, charge and current density. The auxiliary equations relating **D** and **E**,

$$D = \varepsilon E$$

relating **B** and **H**,

$$B = \mu H$$

defining conduction current density,

$$\mathrm{J}_c = \sigma E$$

and defining convection current density in terms of the volume charge density ρ_v

$$\mathrm{J} = v\ \rho_v$$

For ***free space***, where there are no charges ($\rho_v = 0$) and no conduction currents ($\sigma = 0$), Maxwell's equations take the form shown

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \times H = \frac{\partial D}{\partial t}$$

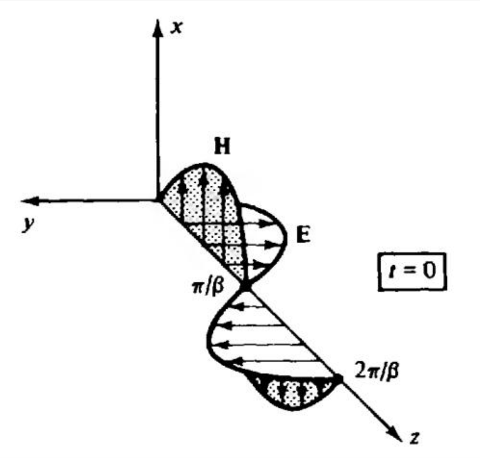
$$\nabla . \mathrm{D} = \rho_\mathrm{v}$$

$$\nabla . \mathrm{B} = 0$$

Example: Given $E = E_m \sin(\omega t - \beta z)\, \mathrm{a}_y$ in free space, find D, B and H?

Solution:

$$D = \varepsilon E = \varepsilon_r \varepsilon_0 E = \varepsilon_0 E_m \sin(\omega t - \beta z)\, \mathrm{a}_y$$



$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix} = -\frac{\partial B}{\partial t}$$

$$-\frac{\partial}{\partial z} E_y \mathbf{a}_x + \frac{\partial}{\partial x} E_y \mathbf{a}_z = -\frac{\partial B}{\partial t}$$

$$-\frac{\partial}{\partial z}(E_m \sin(\omega t - \beta z))\mathbf{a}_x = -\frac{\partial B}{\partial t}$$

$$-\beta E_m \cos(\omega t - \beta z)\, \mathbf{a}_x = -\frac{\partial B}{\partial t}$$

$$B = \int -\beta E_m \cos(\omega t - \beta z)\, dt = \frac{-\beta E_m}{\omega} \sin(\omega t - \beta z)\, \mathbf{a}_x$$

$$H = \frac{B}{\mu} = \frac{-\beta E_m}{\mu \omega} \sin(\omega t - \beta z)\, \mathbf{a}_x$$

Example: Given $H = H_m e^{j(\omega t+\beta z)} \mathrm{a}_x$ in free space, find E?

Solution:

$in\ free\ space\ \sigma = 0$

$$\nabla \times H = \frac{\partial D}{\partial t}$$

$$\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & 0 & 0 \end{vmatrix} = \frac{\partial D}{\partial t}$$

$$\frac{\partial}{\partial z} H_x \mathbf{a}_y = \frac{\partial D}{\partial t}$$

$$\frac{\partial}{\partial z}(H_m)\mathbf{a}_y = \frac{\partial D}{\partial t}$$

$$j\beta H_m e^{j(\omega t+\beta z)}\ \mathbf{a}_y = \frac{\partial D}{\partial t}$$

$$D = \int j\beta H_m e^{j(\omega t+\beta z)}\ dt = \frac{\beta H_m}{\omega} e^{j(\omega t+\beta z)} \mathbf{a}_y$$

$$E = \frac{D}{\varepsilon} = \frac{\beta H_m}{\varepsilon_0 \omega} e^{j(\omega t+\beta z)}\ \mathbf{a}_y$$

Example: Let $\mu = 10^{-5}\ \frac{H}{m}\ ,\ \varepsilon = 4 \times 10^{-9} \frac{F}{m}\ , \sigma = 0\ , and\ \rho_v = 0$ Find $\boldsymbol{k}$ so that each of the following pairs of fields satisfies Maxwell's equations

$(a)\ \ D = 6\ \mathrm{a}_x - 2y\mathrm{a}_y + 2z\mathrm{a}_z \quad ,\ H = \boldsymbol{k}x\ \mathrm{a}_x + 10y\mathrm{a}_y - 25z\mathrm{a}_z$?

Solution:

$$\sigma = 0 \ \ \therefore \mathrm{J}_c = 0 \quad , \quad and\ \frac{\partial D}{\partial t} = 0$$

$$\nabla \times H = 0$$

$$\nabla \times H = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ kx & 10y & -25z \end{vmatrix} = 0$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\frac{\partial B}{\partial t} = \mu \frac{\partial H}{\partial t} = \mu \frac{\partial}{\partial t}\left(\boldsymbol{k}x \, \mathrm{a}_x + 10y\mathrm{a}_y - 25z\mathrm{a}_z\right) = 0$$

$$\nabla \times E = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6 & -2y & 2z \end{vmatrix} = 0$$

$$\nabla . D = \rho_v$$

$$\rho_v = 0$$

$$\nabla . D = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \frac{\partial}{\partial x}6 + \frac{\partial}{\partial y}(-2y) + \frac{\partial}{\partial z}(2z) = 0$$

$$\nabla . B = 0$$

$$\frac{\partial}{\partial x}(kx) + \frac{\partial}{\partial y}(10y) + \frac{\partial}{\partial z}(-25z) = 0$$

$$k + 10 - 25 = 0$$

$$k = 15$$

$(b)\ E = (20y - kt)\,\mathrm{a}_x \qquad , \ H = (y + 2 \times 10^6 t)\mathrm{a}_z$?

$$\nabla \times H = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & H_z \end{vmatrix} = \frac{\partial H_z}{\partial y}\mathbf{a}_x = \frac{\partial}{\partial y}(y + 2 \times 10^6 t)\mathbf{a}_x = \mathbf{1}$$

$$\frac{\partial D}{\partial t} = \varepsilon \frac{\partial E}{\partial t} = \varepsilon \frac{\partial}{\partial t}(20y - kt) = -\varepsilon k$$

$$-\varepsilon k = 1$$

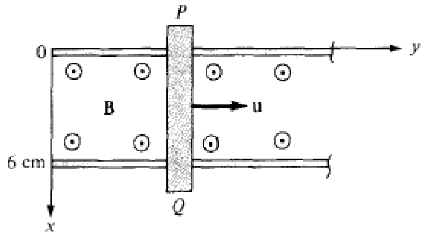
$$k = \frac{-1}{\varepsilon} = \frac{-1}{4 \times 10^{-9}} = -2.5 \times 10^8$$

Homework

Q₁: Given $H = 300\cos(3\times10^8\, t - y)\mathrm{a}_z\, A/m$ in free space, find the emf developed in the general $\mathrm{a}_\emptyset$ direction about the closed path having corners at (a) (0, 0, 0), (1, 0, 0), (1, 1, 0), and (0, 1, 0); (b) (0, 0, 0) (2π, 0, 0), (2π, 2π, 0), and (0, 2π, 0).?

Ans: $-1.13\times10^5\,[\cos(3\times10^8\, t - 1) - \cos(3\times10^8\, t)]$, *0*

Q₂: A conducting bar can slide freely over two conducting rails as shown in Figure below. Calculate the **induced voltage** in the bar



(a) If the bar is stationary at **y = 8** cm and $\boldsymbol{B = 4\cos(10^6 t)\,\mathrm{a}_z}$

(b) If the bar slides at a velocity $\boldsymbol{v = 20\mathrm{a}_y}$ m/s and $\boldsymbol{B = 4\mathrm{a}_z}$

(c) If the bar slides at a velocity $\boldsymbol{v = 20\mathrm{a}_y}$ m/s and $\boldsymbol{B = 4\cos(10^6 t - y)\,\mathrm{a}_z}$

Q₃: Find the constant **a**, **b** and **c** so that the following fields satisfies Maxwell's equations in free space, $B = (\mathrm{a}\, y^2 + bt^2)\mathrm{a}_z$ $\quad E = cyt\,\mathrm{a}_x$

Q₄: Determine whether or not the following fields satisfy Maxweel's equations for which $\varepsilon_r = 8\;, \mu_r = 2\;\; and\;\; \rho_v = 0$,

$(a)\; B = 4\mathrm{a}_x \qquad E = 2\,\mathrm{a}_y$

$(b)\; H = \frac{1}{\pi}\cos(3\times10^8\, t - 4x)\mathrm{a}_z \qquad E = 60\cos(3\times10^8\, t - 4x)\mathrm{a}_y$

Q₅: Given $H = 10^{-5}\rho\cos(\omega\, t)\mathrm{a}_\emptyset$ in free space, find J_D?

Q₆: *A vector potential is given as* $A = A_0\cos(\omega\, t - kz)\mathrm{a}_y$,$\sigma = 0$ *find H and E?*